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Conflict & Cooperation in Symmetric Potential Games

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Abstract In this paper we consider a special class of n-person potential games and investigate partial cooperation between a portion of the players that sign a cooperative agreement. Existence results of partial cooperative equilibria are obtained and some applications are discussed, particularly an international water resources management model.

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1 Introduction

In various practical situations the interaction between agents can be a mixture of non-cooperative and cooperative behavior. In general the study of this partial cooperation is difficult to manage with existing game theory models. That is why we restrict in this paper to special symmetric conflict situations where finding pure Nash equilibria is an optimization problem. It turns out that still in these restrictive situations some interesting practical cases can be handled. To be more precise our starting point is a subclass of symmetric strategic games with a potential. Such potential games were introduced by Monderer and Shapley (1996). Under some extra conditions such games possess pure symmetric Nash equilibria which can be obtained by looking at symmetric strategy profiles where the corresponding potential function is maximal. Also reduced games arising from a merge of a fraction of the players turn out to be of the same type. The outline of the paper is as follows. In Section 2 symmetric potential games are considered and sufficient conditions are given to guarantee the existence of pure symmetric Nash equilibria. Also the notion of cooperative equilibrium is introduced and the existence of such equilibria for symmetric potential games is discussed. In Section 3 attention is paid to partial cooperation and the existence of partial cooperative equilibria. Also comparisons are made between the payoffs given different intensities of partial cooperation. In Section 4 our results are applied to symmetric Cournot games and in Section 5 to international water resource management situations.

2 Conflict & Cooperation

Let $\Gamma =< n; X; f_1, ..., f_n >$ be an n-person normal form game with player set $I = \{1, 2, ..., n\}$, with the same strategy space X for each player $i \in I$ and where $f_i \colon X^n \mapsto \mathcal{R}$ is the payoff function of player $i \in I$. If player i chooses $x_i \in X$, then he obtains a profit $f_i(x_1, ..., x_n)$. Each player wants to maximize his own profit. We denote by x_{-i} the vector $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X^{n-1}$.

If noncooperative behavior is assumed between the n players, the equilibrium solution considered is the well known concept of Nash equilibrium.

A Nash equilibrium is a vector $x^N = (x_1^N, ..., x_n^N) \in X^n$ such that for each $i \in I$

$$f_i(x_1^N, \ldots, x_n^N) = \max_{y \in X} f_i(x_1^N, \ldots, x_{i-1}^N, y, x_{i+1}^N, \ldots, x_n^N).$$

Let us suppose that $\Gamma =< n; X; P, h >$ is a *symmetric exact potential game* or *symmetric potential game*, i.e. there exists a potential function

$P\colon X^n \mapsto \mathcal{R}$ and a function $h\colon X^{n-1} \mapsto \mathcal{R}$ such that, for all i

$$f_i(x_1, \ldots, x_n) = P(x_1, \ldots, x_n) + h(x_{-i})$$

and P is a symmetric function, i.e. $P(x) = P(x')$ for all permutations x' of the vector $x \in X^n$. The definition of exact potential game has been given by Monderer and Shapley (1996).

Recall that $\Gamma =< n; X; f_1, ..., f_n >$ is called an *exact potential game* if there is a potential function $P: X^n \mapsto \mathcal{R}$ such that for all $i \in I$ and for any $x_{-i} \in X^{n-1}$

$$f_i(y_i, x_{-i}) - f_i(z_i, x_{-i}) = P(y_i, x_{-i}) - P(z_i, x_{-i})$$

for all $y_i, z_i \in X$. The game $\Gamma =< n; X; f_1, ..., f_n >$ is called an *ordinal potential game* if there is a potential function $P: X^n \mapsto \mathcal{R}$ such that for all $i \in I$ and for any $x_{-i} \in X^{n-1}$

$$f_i(y_i, x_{-i}) - f_i(z_i, x_{-i}) > 0 \Longleftrightarrow P(y_i, x_{-i}) - P(z_i, x_{-i}) > 0$$

for all $y_i, z_i \in X$. Clearly, the equilibrium set of an ordinal potential game $\Gamma =< n; X; f_1, ..., f_n >$ coincides with the the equilibrium set of the n-person game $\Gamma =< n; X; P, ..., P >$, being P the potential function. Consequently, elements of argmax(P) are Nash equilibria of the game.

Let X be a convex subset of an Euclidean space and let $f : X \mapsto \mathcal{R}$. We say that f is concave if for any $x, y \in X$ and for any $\alpha \in (0, 1)$ we have that $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$ and that f is strictly concave if for any $x, y \in X$, with $x \neq y$, and for any $\alpha \in (0, 1)$ we have that $f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y)$. We say that f is quasi-concave if for any $x, y \in X$ and for any $\alpha \in [0, 1]$ we have that $f(\alpha x + (1-\alpha)y) \geq min[f(x), f(y)]$. If the f is a strictly concave function, it is also quasi concave. By using the Nash's theorem for symmetric games (Moulin), we have the following existence result.

Proposition 2.1 If X is a closed real interval, P is a continuous function quasi-concave in x_i for all $i = 1, ..., n$, there exists at least one symmetric Nash equilibrium (i.e. a Nash equilibrium with identical components) of the symmetric potential game $\Gamma =< n; X; P, h >$.

For simplicity we will denote $(y, ..., y) \in X^k$ by $y_{\mathbf{k}}$, for all $y \in X$ and $k = 2, ..., n$. When a symmetric Nash equilibrium is chosen $x_1^N = x_2^N = ... = x_n^N = \eta^N$, each player receives the same payoff $V^N = f_i(\eta^N_{\mathbf{n}}) = P(\eta^N_{\mathbf{n}}) + h(\eta^N_{\mathbf{n-1}})$ for any $i \in I$, and the total benefit is nV^N.

A symmetric equilibrium can be also reached if all players sign a cooperative agreement and decide to choose the same strategy, i.e. they look for an equilibrium with the condition $x_1 = x_2 = ... = x_n = y$ for $y \in X$. In this case we have $f_i(y_{\mathbf{n}}) = P(y_{\mathbf{n}}) + h(y_{\mathbf{n-1}})$ for all $i \in I$.

Definition 2.1 A *cooperative equilibrium* is a vector $x^C = (\xi^C_{\mathbf{n}}) \in X^n$ such that

$$\xi^C \in \underset{y \in X}{\operatorname{argmax}} \left(P(y_{\mathbf{n}}) + h(y_{\mathbf{n-1}}) \right).$$

The existence of at least one cooperative equilibrium of the symmetric potential game $\Gamma =< n; X; P, h >$, is guaranteed if P and h are upper semicontinuous functions and X a closed real interval. In the case where the players agree for a cooperative equilibrium, they receive a payoff $V^C = P(\xi^C_{\mathbf{n}}) + h(\xi^C_{\mathbf{n-1}})$ and the total benefit is nV^C.

Remark 2.1 Suppose that the assumptions of Proposition 2.1 hold and that h is an upper semicontinuous function on X^{n-1}. Let $x^N = (\eta^N_{\mathbf{n}}) \in X^n$ be a symmetric Nash equilibrium and $x^C = (\xi^C_{\mathbf{n}}) \in X^n$ a cooperative equilibrium. Since $f_i(\xi^C_{\mathbf{n}}) \geq f_i(y_{\mathbf{n}})$ for all $y \in X$, for $y = \eta^N$, we have

$$V^C = f_i(\xi^C, ..., \xi^C) \geq f_i(\eta^N, ..., \eta^N) = V^N.$$

Example 2.1 Let us consider $n = 2$, $X = \{a, b, c\}$ and the following payoffs

	a	b	c
a	$0, 0$	$\frac{1}{2}, -\frac{1}{4}$	$1, -1$
b	$-\frac{1}{4}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}$	$\frac{3}{4}, -\frac{1}{2}$
c	$-1, 1$	$-\frac{1}{2}, \frac{3}{4}$	$0, 0$

The game is a symmetric (exact) potential game with symmetric potential

	a	b	c
a	0	$-\frac{1}{4}$	-1
b	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{5}{4}$
c	-1	$-\frac{5}{4}$	-2

Here (a, a) is a Nash equilibrium and $V^N = 0$. Let us suppose that both players sign an agreement. Then the reduced game (diagonal game) is

	a	b	c
a	$0,0$	$-,-$	$-,-$
b	$-,-$	$\frac{1}{4},\frac{1}{4}$	$-,-$
c	$-,-$	$-,-$	$0,0$

The cooperative equilibrium is (b,b) with $b = \underset{X}{\text{argmax}}\{0, 1/4, 0\}$ and $V^C = 1/4 > 0 = V^N$.

Remark 2.2 A cooperative agreement will not be sustainable if a symmetric Nash equilibrium does not exist. Let us consider $n = 2$, $X = \{a, b\}$ and the following payoffs

	a	b
a	$2,2$	$4,4$
b	$4,4$	$2,2$

This game is a symmetric potential game with potential

	a	b
a	2	4
b	4	2

and $h = 0$, and there are two Nash equilibria (a, b) and (b, a). If both players decide to cooperate they choose one of the two cooperative equilibria (a, a) and (b, b), but they receive a lower profit.

3 Partial Cooperation

In different situations, for example in Beaudry *et al.* (2000), in Becker and Easter (1999), a portion of the n players may sign a cooperative agreement. Let $P_{k+1}, ..., P_n$ be the players acting in a cooperative way and $P_1, ..., P_k$ the players acting in a noncooperative way, for $k = 0, ..., n$. Now we define a concept of equilibrium concerning this partial cooperation situation. We assume the last $n - k$ players (cooperating players) use the same strategy, i.e. $x_{k+1} = x_{k+2} = ... = x_n = y$ for $y \in X$. The first k players with payoffs

$$f_i(x_1, ..., x_k, y_{\mathbf{n-k}}) = P(x_1, ..., x_k, y_{\mathbf{n-k}}) + h(x_1, .., x_{i-1}, x_{i+1}, .., x_k, y_{\mathbf{n-k}})$$

for any $i = 1, ..., k$ do not cooperate and choose a Nash equilibrium $(x_1^N(y), ..., x_k^N(y)) \in X^k$ for any $y \in X$.

For all $k = 2, ..., n$ and for all $y \in X$, let us consider the normal form game $\Gamma_k(y) =< k; X; P(\cdot, y), h(\cdot, y) >$, i.e. the k-person game with the same strategy space X for each player and payoff function of player i, for $i = 1, ..., k$, given by $f_i(x_1, ..., x_k, y_{\mathbf{n-k}})$ for any $y \in X$.

Proposition 3.1 If $\Gamma =< n; X; P, h >$ is a symmetric potential game, the game $\Gamma_k(y) =< k; X; P(\cdot, y), h(\cdot, y) >$ is also a symmetric potential game for all $k = 2, ..., n$ and for all $y \in X$, being $P(x_1, ..., x_k, y_{\mathbf{n-k}})$ the potential function.

Proof. For any $i = 1, ..., k$ the function

$$f_i(x_1, ..., x_k, y_{\mathbf{n-k}}) - P(x_1, ..., x_k, y_{\mathbf{n-k}}) = h(x_1, .., x_{i-1}, x_{i+1}, .., x_k, y_{\mathbf{n-k}})$$

does not depend on x_i and by using the characterization by Facchini *et al.* (1997) the game is a potential game. Moreover P is a symmetric function.

We suppose that the game $\Gamma_k(y) =< k; X; P(\cdot, y), h(\cdot, y) >$, for all $k = 2, ..., n$, has a unique symmetric Nash equilibrium $(\eta_{\mathbf{k}}^N(y))$ for all $y \in X$. Then, for all $y \in X$, each of the last $n - k$ players (cooperating players) has the following payoff

$$f_i(\eta_{\mathbf{k}}^N(y), y_{\mathbf{n-k}}) = P(\eta_{\mathbf{k}}^N(y), y_{\mathbf{n-k}}) + h(\eta_{\mathbf{k}}^N(y), y_{\mathbf{n-k-1}})$$

for $i = k + 1, ..., n$.

Definition 3.1 A vector $x^P = (\eta_{\mathbf{k}}^N(\xi^C), \xi_{\mathbf{n-k}}^C) \in X^n$ such that

$$\xi^C \in \operatorname*{argmax}_{y \in X} \left(P(\eta_{\mathbf{k}}^N(y), y_{\mathbf{n-k}}) + h(\eta_{\mathbf{k}}^N(y), y_{\mathbf{n-k-1}}) \right)$$

is called a *partial cooperative equilibrium* of the game $\Gamma =< n; X; P, h >$. The fully cooperative solution of Definition 2.1 is given by $k = 0$ while the fully noncooperative solution (Nash equilibrium solution) is given by $k = n$. The following existence result for partial cooperative equilibria holds.

Proposition 3.2 Let X be a closed real interval, P a continuous and strictly concave function on X^n, h a upper semicontinuous function on X^{n-1}. For all $k = 2, ..., n$, the symmetric potential game $\Gamma =< n; X; P, h >$ has at least a partial cooperative equilibrium $x^P = (\eta^N_{\mathbf{k}}(\xi^C), \xi^C_{\mathbf{n-k}}) \in X^n$.

Proof. For any fixed $k > 1$, for any $y \in X$ the strictly concave potential function $P(x_1, ..., x_k, y_{\mathbf{n-k}})$ has a unique maximum point that is also a Nash equilibrium point of the game $\Gamma_k(y) =< k; X; P(\cdot, y), h(\cdot, y) >$. Since P is a symmetric function, the unique Nash equilibrium $(\eta^N_{\mathbf{k}}(y)) \in X^k$ is symmetric too. Since X is compact and P is a continuous function, by using the Berge's theorem (Border, 1985), we have that the function $y \mapsto \eta^N(y) \in X$ turns out to be continuous on X and $F(y) = P(\eta^N_{\mathbf{k}}(y), y_{\mathbf{n-k}}) + h(\eta^N_{\mathbf{k-1}}(y), y_{\mathbf{n-k}})$ to be upper semicontinuous function on X.

Remark 3.1 In order to obtain the continuity of the function $y \mapsto \eta^N(y) \in X$, we can also assume in Proposition 3.2 that a unique Nash equilibrium of $\Gamma_k(y)$ exists for any $y \in X$ and all $k = 2, ..., n$, instead of the assumption of strictly concavity of the potential function P.

In the partial cooperative equilibrium $x^P = (\eta^N_{\mathbf{k}}(\xi^C), \xi^C_{\mathbf{n-k}}) \in X^n$, the total benefit will be $V^P(k) = kV^N_k + (n-k)V^C_{n-k}$ where

$$V^N_k = P(\eta^N_{\mathbf{k}}(\xi^C), \xi^C_{\mathbf{n-k}}) + h(\eta^N_{\mathbf{k-1}}(\xi^C), \xi^C_{\mathbf{n-k}})$$

$$V^C_{n-k} = P(\eta^N_{\mathbf{k}}(\xi^C), \xi^C_{\mathbf{n-k}}) + h(\eta^N_{\mathbf{k}}(\xi^C), \xi^C_{\mathbf{n-k-1}}).$$

In the fully cooperative solution case $V^P(0) = nV^C$ and in the fully noncooperative solution case $V^P(n) = nV^N$. The total benefit $V^P(k)$ is an intermediate value between nV^N and nV^C and decreasing with the number k of player who do not cooperate. In fact for any fixed $k = 2, ..., n-1$ and for any $y \in X$ we have $h(\eta^N_{\mathbf{k-1}}(y), y_{\mathbf{n-k}}) \leq h(\eta^N_{\mathbf{k}}(y), y_{\mathbf{n-k-1}})$ being $(\eta^N_{\mathbf{k}}(y))$ a Nash equilibrium of the game $\Gamma_k(y)$, so $V^N_k \leq V^C_{n-k}$.

Example 3.1 Let us consider $n = 4$, $X = [0, 1]$ and the following payoffs

$$f_i(x_1, x_2, x_3, x_4) = -x_i^2 + \sum_{j \neq i} x_j, \; i = 1, 2, 3, 4.$$

The game is a symmetric potential game with $P(x_1, x_2, x_3, x_4) = -x_1^2 - x_2^2 - x_3^2 - x_4^2$ and $h(x_{-i}) = \sum_{j \neq i} x_j^2 + x_j$ for $i = 1, 2, 3, 4$, the Nash equilibrium is $x^N = (0, 0, 0, 0)$ and $V^N = 0$; the total benefit is $nV^N = 0$.

Let $x_1 = x_2 = x_3 = x_4 = y$, the problem $\max_{y \in [0,1]} -y^2 + 3y$ has solution $\xi^C = 1$ and the cooperative equilibrium in this example is $x^C = (1, 1, 1, 1)$, $V^C = 2$ and the total benefit $nV^C = 8$.

We consider the partial cooperative situation where two of the four players cooperate. For any $y \in [0, 1]$, the two-player game with payoffs

$$f_1(x_1, x_2, y, y) = -x_1^2 + x_2 + 2y$$

$$f_2(x_1, x_2, y, y) = -x_2^2 + x_1 + 2y$$

has one equilibrium $(0, 0)$; the last two cooperating players have to solve the problem

$$\max_{y \in [0,1]} -y^2 + y$$

and the partial equilibrium is $x^P = (0, 0, 1/2, 1/2)$ with total benefit $V^P(2) = 5/2$.

4 Symmetric Cournot Games

Consider an n-firm competitive market of a single homogeneous commodity with linear inverse demand function $\mathcal{P}$. Suppose that any firm i can supply the single product in any non negative bounded quantity $q_i \in [0, q^0]$ ($q^0 > 0$) and they have the same cost c per unit produced. If $Q = \sum_{i=1}^n q_i$ is the aggregate quantity, we suppose that the inverse demand is $\mathcal{P}(Q) = \alpha - \beta Q$, with $\alpha > c \geq 0, \beta > 0$. The objective of firm i is to maximize its profit

$$\Pi_i(q_1, ..., q_n) = q_i(\alpha - \beta \sum_{i=1}^n q_i) - cq_i.$$

The n-person normal form game $\Gamma =< n; X; \Pi_1, ..., \Pi_n >$, with $X = [0, q^0]$ is called a *quasi Cournot game* (Monderer and Shapley, 1996).

As observed in MasColell *et al.* (1995, pag 392), in the case of a duopoly game, there exists a unique symmetric Nash equilibrium $q_1^N = q_2^N = \frac{1}{3}[(\alpha - c)/\beta]$ with total benefit $nV^N = 2\frac{1}{9}[(\alpha - c)^2/\beta]$. The symmetric joint monopoly point $q_1^C = q_2^C = \frac{1}{4}[(\alpha - c)/\beta]$ (cooperative equilibrium), at which each firm produces half of the monopoly output of $\frac{1}{2}[(\alpha - c)/\beta]$, is each firm's most profitable point on the ray $q_1 = q_2$. In fact, the total benefit in the cooperative case is $nV^C = 2\frac{3}{8}[(\alpha - c)^2/\beta] > nV^N$.

Analogous arguments can be used in the general oligopoly case $n > 2$ because a quasi Cournot game is an exact potential game with potential function

$$P(q_1, ..., q_n) = (\alpha - c) \sum_{i=1}^n q_i - \beta \sum_{i=1}^n q_i^2 - \beta \sum_{1 \leq i < j \leq n} q_i q_j.$$

Because of the constant marginal cost c of each firm, it is also a symmetric potential game with

$$h(q_{-i}) = -(\alpha - c) \sum_{1 \leq j \leq n, j \neq i} q_j + \beta \sum_{1 \leq j \leq n, j \neq i} q_j^2 + \beta \sum_{1 \leq l < j \leq n, l, j \neq i} q_l q_j.$$

Under the above assumptions on the inverse demand function and on the cost functions, the game $\Gamma =< n; X; \Pi_1, ..., \Pi_n >$ has a unique Nash equilibrium point. For example, the sufficient conditions by Vives (1999, pag. 98) are satisfied. In fact $\mathcal{P}$ is a decreasing function and $\mathcal{P}$ is a log-concave function, i.e. $\log(\mathcal{P})$ is concave. This unique Nash equilibrium point is symmetric and by using Proposition 3.2 and Remark 3.1, we have the existence of a partial cooperative equilibrium for any $k = 2, ..., n$, being $n - k$ the cooperating firms. The same holds if all firms have the same convex cost function $c(\cdot)$.

5 International Water Resources Management Model

As in the paper by Becker and Easter (1999), let $\Gamma =< n; A; \Pi_1, ..., \Pi_n >$ be an n-person game with strategy space $A = [0, \hat{D}]$ and $\Pi_i \colon A^n \mapsto \mathcal{R}^+$, $i \in I$, the payoff function of player i. If player i diverts an amount of water $d_i \in A$ from an international water body for all $i \in I$, then he obtains a net benefit $\Pi_i(d_1, ..., d_n)$, where

$$\Pi_i(d_1, ..., d_n) = B_i(d_i) - C_i(d_1 + ... + d_n)$$

being B_i the benefit to user i from diverting d_i and C_i the cost for user i. Each user's cost depends on the total amount diverted $d_1 + ... + d_n$ and include both the direct cost of diverting water as well as the indirect cost associated with the decreasing lake levels.

It is easy to see that if the players have identical costs functions, i.e. $C_i = C$ for all $i \in I$, the game $\Gamma =< n; A; \Pi_1, ..., \Pi_n >$ is a symmetric potential game with potential function

$$P(d_1, ..., d_n) = \sum_{i=1}^{n} B_i(d_i) - C(d_1 + ... + d_n)$$

and $h(d_{-i}) = -\sum_{1 \leq j \leq n, j \neq i} B_j(d_j)$.

Let $\sigma = \frac{k}{n}$ be a portion of players, $k = 0, 1, ..., n$. Suppose that $(1 - \sigma)n$ is the number of users who sign a cooperative agreement and form a coalition, while σn is the number of users who do not sign the agreement. The fully cooperative solution is given by $\sigma = 0$ while the fully noncooperative solution is given by $\sigma = 1$.

For any σ, let $\{P_1, ..., P_{\sigma n}\}$ be the set of players acting in a noncooperative way and $\{P_{\sigma n+1}, ..., P_n\}$ the set of players acting in a cooperative way. We assume the last $(1-\sigma)n$ players act identically, i.e. $d_{\sigma n+1} = d_{\sigma n+2} = ... = d_n = d$. For all $d \in A$, the first σn players with payoffs

$$\Pi_i(d_1, ..., d_{\sigma n}, d_{\mathbf{(1-\sigma)n}}) = B_i(d_i) - C(d_1 + ... + d_{\sigma n} + n(1-\sigma)d), \; i = 1, ..., \sigma n$$

choose a Nash equilibrium $(\bar{d}_1(d), ..., \bar{d}_{\sigma n}(d)) \in A^{\sigma n}$.

The last $(1-\sigma)n$ players have net benefit equal to

$$\Pi_i(\bar{d}_1(d), ..., \bar{d}_{\sigma n}(d), d_{\mathbf{(1-\sigma)n}}) = B_i(d) - C(\bar{d}_1(d) + ... + \bar{d}_{\sigma n}(d) + (1-\sigma)nd)$$

for $i = \sigma n + 1, ..., n$. The equilibrium for them is $(\bar{d}_{\mathbf{(1-\sigma)n}}) \in A^{(1-\sigma)n}$ such that

$$\bar{d} \in \underset{d \in A}{\operatorname{argmax}} \left(B_i(d) - C(\bar{d}_1(d) + ... + \bar{d}_{\sigma n}(d) + (1-\sigma)nd) \right)$$

The vector $(\bar{d}_1(\bar{d}), ..., \bar{d}_{\sigma n}(\bar{d}), \bar{d}_{\mathbf{(1-\sigma)n}}) \in A^n$ is a partial cooperative equilibrium of the game $\Gamma =< n; A; \Pi_1, ..., \Pi_n >$, being $(1-\sigma)n$ the number of cooperating users.

Becker and Easter (1999) considered this model with a positive increasing continuous and concave quadratic benefit function $B_i(d_i) = b_1 d_i - b_2 d_i^2/2$ $(b_1 > 0, b_2 > 0)$ and a positive increasing continuous and convex quadratic cost function $C(d_1 + ... + d_n) = c_1(d_1 + ... + d_n) + c_2(d_1 + ... + d_n)^2/2$ $(c_1 > 0, c_2 > 0)$. All assumptions of Proposition 3.2 are satisfied and we have the existence of a partial cooperative equilibrium for any $k = 2, ..., n$, being $n-k$ the cooperating users. In this case the partial cooperative equilibrium is derived explicitly and some numerical data concerning an analysis of Lake Ontario are discussed.

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