The 6th MEETING ON GAME THEORY AND PRACTICE
Zaragoza, Spain 10-12 July

NASH EQUILIBRIUM IN BIORESOURCE MANAGEMENT PROBLEM WITH CHANGING AREA FOR FISHERY

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Abstract A dynamic game model of bioresource management problem (fish catching) is considered. The center (state), which determines a share of prohibited for catching (reserved) area of a reservoir, and the players (fishing firms), which make fish catching are the participants of this game. Each player is independent decision maker, being guided by maximization of the profit of fish sale. In traditional statement the center's objective is catch regulation by introduction quotas on fishing. In this paper the center's task in each moment is the choice of optimal share of reserved territory for maintenance of stable population development in a reservoir in long-term prospect and possible fishing level's definition, sufficient for demand satisfaction.

JEL (or AMS) references: 91B76, 91B62

Key Words: Nash equilibrium, bioresorce management, fishery, reserved area.

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1. How to determine the reserved area

We consider a center which has to determine a reserved area in the lake or the sea, called $s(t)$, $0 \le s(t) \le 1$. The part used for fishing is $1 - s(t)$. The time horyzon is $[0, T]$. Let the population has the dynamics

$$x'(t) = F(x(t)) - qE(t)(1 - s(t))x(t),\ 0 \le t \le T,\ x(0) = x_0\,, \tag{1.1}$$

where $x(t) \ge 0$ – size of populatin at the moment t; F – function of development; $E(t) \ge 0$ – fisheries efforts at the moment t, $s(t)$ – the part of the area prohibited for fishery and $q > 0$ – catch coefficient for a unit of efforts.

Suppose the dynamics is determined by Verhulst model:

$$F(x) = rx(1 - x/K)\,.$$

Then the reward of the player is

$$J = g(x(T)) + \int_0^T e^{-\rho t}[\Pi(q, s(t), x(t), E(t)) \cdot qE(t)(1 - s(t))x(t) - c^0 E(t)]dt\,, \tag{1.2}$$

where ρ – discount factor, c^0 – cost for fishing and Π – price function

$$\Pi(q, s(t), x(t), E(t)) = p - kqE(t)(1 - s(t))x(t),\ p, k > 0\,.$$

Here $g(x)$ is the future reward at the final moment T $g'(x) \ge 0,\ g''(x) \le 0\,.$ Let $g(x(t)) = \alpha p x(t)e^{-\rho t}$.

Rewrite the reward of the player as:

$$J = g(x(T)) + \int_0^T [-\frac{1}{2}aE(t)^2(1 - s(t))^2 x(t)^2 + bE(t)(1 - s(t))x(t) - cE(t)]dt\,,$$

where $a = 2kq^2 \exp(-\rho t)\,, b = pq \exp(-\rho t)\,, c = c^0 \exp(-\rho t)\,.$

Consider a payoff of the center in the form

$$I_1 = -\int_0^T (x(t) - \bar{x}(t))^2 dt\,, \tag{1.3}$$

where $\bar{x}(t)$ – is the size of population optimal for reproduction.

We search the optimal solution:

$$\begin{cases} J(s^*, E^*) \ge J(s^*, E)\,,\ \forall E\,, \\ I_1(s^*, E^*) \ge I_1(s, E^*)\,,\ \forall s\,. \end{cases} \tag{1.4}$$

Notice at the traditional approach [2-4] the objective of the center is the determination of quota for fisheries. In [5-7] the models with reserved area were investigated for different scenarios. But there the center determines a part of reserved area once and after that it doesn't change it. In this paper the strategy of the center is a function $s(t)$, and the solution is a Nash equilibrium.

Let us fix a strategy of the player (fishing farm) and find the optimal behavior of the center. Hamilton function [8] for the leader is

$$H_1 = -(x-\bar{x})^2 + \lambda_1(F(x) - qE(1-s)x).$$

Presenting $H_1 = \lambda_1 qExs +$ $\{$ part not depending on $s\}$, we obtain the strategy s miximizing H_1:

$$\begin{cases} \text{If } \lambda_1 > 0 \Longrightarrow s(t) \equiv 1\,, \\ \text{if } \lambda_1 = 0 \Longrightarrow s(t) \text{ any}\,, \\ \text{if } \lambda_1 < 0 \Longrightarrow s(t) \equiv 0\,. \end{cases}$$

According to the Maximum principle the condition for λ_1 is

$$\lambda_1' = -\frac{\partial H_1}{\partial x} = 2(x-\bar{x}) - \lambda_1(F_x'(x) - qE(1-s))\,,\ \ \lambda_1(T) = 0\,.$$

Now we fix a strategy of the center. Then the Hamiltonian for the player is:

$$H_2 = -\frac{1}{2}aE^2(1-s)^2x^2 + bE(1-s)x - cE + \lambda_2(F(x) - qE(1-s)x)\,.$$

Find maximum of H_2

$$\frac{\partial H_2}{\partial E} = -aE(1-s)^2x^2 + b(1-s)x - c - \lambda_2 q(1-s)x = 0\,.$$

It yields

$$E(t) = \left[\frac{(b-q\lambda_2)(1-s)x - c}{a(1-s)^2x^2}\right]^+\,.$$

Control function of the player is non-negative if

$$(b-q\lambda_2)(1-s)x \geq c\,. \tag{1.5}$$

The equation for λ_2 and transversability condition give

$$\lambda_2' = -\frac{\partial H_2}{\partial x} = -E(1-s)(b - aE(1-s)x) - \lambda_2(F_x' - qE(1-s))\,,\ \ \lambda_2(T) = g_x'(x^*(T))\,.$$

We will construct the optimal behavior of the center in piecewise continuous form.

1.1. Special cases

Evidently, that the optimal control s^* has satisfied $x(t) \equiv \bar{x}$. But at the initial moment it couldn't be satisfied. So, there are possible two variants.

1) If $x(0) < \bar{x}$, then the control is equal to 1 (no fishery) and then s^* such that $x(t) \equiv \bar{x}$ for $t > t_0$.

2) If $x(0) > \bar{x}$, then the control is equal to 0 (no limitation for fishery) and then s^* such that $x(t) \equiv \bar{x}$ for $t > t_0$.

Theorem 1. *Suppose $x(0) < \bar{x}$.*

Nash equlibrium in the problem (1.4) is:

$$s^*(t) = \begin{cases} 1\,, & t < t_0 \\ \bar{s}(t)\,, & t \geq t_0 \end{cases} \qquad E^*(t) = \begin{cases} 0\,, & t < t_0 \\ \bar{E}(t)\,, & t \geq t_0\,, \end{cases}$$

where

$$\bar{s}(t) = 1 - \frac{c_0 K}{\bar{x}q(K(p - \bar{\lambda}_2(t)) - 2rk\bar{x}(K - \bar{x}))}\,, \quad \bar{E}(t) = \frac{(p - \bar{\lambda}_2(t))q\bar{x}(1 - \bar{s}(t)) - c_0}{2kq^2\bar{x}^2(1 - \bar{s}(t))^2}\,,$$

$$t_0 = \frac{1}{r}\ln\frac{\bar{x}(K - x_0)}{x_0(K - \bar{x})}\,,$$

and $\bar{s}(t)$ satisfies (1.6) and

$$\bar{\lambda}_2(t) = \frac{(K - \bar{x})^2(\frac{Kpr}{K - \bar{x}} - 2r^2\bar{x}k)(1 - e^{\frac{(t-T)(r\bar{x}+K\rho)}{K}}) + \alpha p K(r\bar{x} + K\rho)e^{\frac{(t-T)(r\bar{x}+K\rho)}{K}}}{K(r\bar{x} + K\rho)}$$

satisfies:

$$M - \frac{p}{2} < \bar{\lambda}_2(t) < min\{\frac{p}{4}, M - \frac{c_0}{\bar{x}q}\}\,, \tag{1.7}$$

with $M = p - \frac{2rk\bar{x}}{K}(K - \bar{x})$.

Let, for example, $\bar{x} = 250000$ and initial size of population is $x(0) = 150000$, the moment $t_0 = 26.82$ after that $\lambda_1(t) \equiv 0$ (fig. 1.3). Consequently, at the begining $s^* = 1$, and then $s^* = \bar{s}(t)$ (fig. 1.4). The optimal value $E^*(t)$ you can see in fig. 1.2. The size of population increases from 150000 to 250000 and then stay at this level (fig. 1.1).

Let us check the conditions for $\bar{\lambda}_2(t)$. For these parameters $M = 2000$, $M - \frac{p}{2} = -1000$, $min\{\frac{1}{4}p, M - \frac{c_0}{\bar{x}q}\} = 1000$. So, we obtain $-1000 < \bar{\lambda}_2(t) < 1000$. By simulations we obtain that $\bar{\lambda}_2(t)$ is increasing function, $\bar{\lambda}_2(0) = 285$ and $\bar{\lambda}_2(T) = 300$. So, the conditions of the Theorem are satisfied.

The loss of the center is $I_1 = 0.829\, \cdot\, 10^{11}$.

Theorem 2. *Suppose $x(0) > \bar{x}$. Nash equlibrium in the problem (1.4) is:*

$$s^*(t) = \begin{cases} 0\,, & t < t_0 \\ \bar{s}(t)\,, & t \geq t_0 \end{cases} \qquad E^*(t) = \begin{cases} \frac{(p - \hat{\lambda}_2(t))qx(t) - c_0}{2kq^2x(t)^2}\,, & t < t_0 \\ \bar{E}(t)\,, & t \geq t_0\,, \end{cases} \tag{1.11}$$

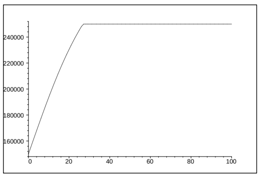

Fig. 1.1. Value $x^*(t)$

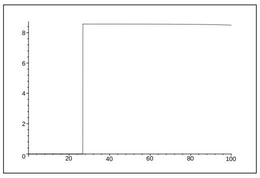

Fig. 1.2. Value $E^*(t)$

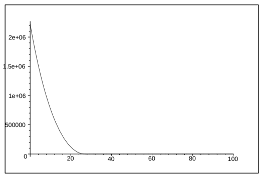

Fig. 1.3. Value $\lambda_1(t)$

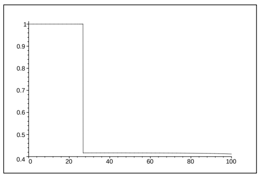

Fig. 1.4. Value $s^*(t)$

where $x(t)$ and $\hat{\lambda}_2(t)$ are determined by system (1.13) *and satisfy*

$$x(t) > \frac{2c_0}{pq}\,,\ \ \hat{\lambda}_2(t) < \frac{p}{4}\,,$$

$$\bar{s}(t) = 1 + \frac{c_0 K}{\bar{x}q(2rk\bar{x}(K-\bar{x}) - K(p-\bar{\lambda}_2(t)))}\,,\ \ \bar{E}(t) = \frac{(p-\bar{\lambda}_2(t))q\bar{x}(1-\bar{s}(t)) - c_0}{2kq^2\bar{x}^2(1-\bar{s}(t))^2}\,,$$

and $\bar{s}(t)$ satisfies (1.6), and

$$\bar{\lambda}_2(t) = \frac{(K-\bar{x})^2(\frac{Kpr}{K-\bar{x}} - 2r^2\bar{x}k)(1-e^{\frac{(t-T)(r\bar{x}+K\rho)}{K}}) + \alpha pK(r\bar{x}+K\rho)e^{\frac{(t-T)(r\bar{x}+K\rho)}{K}}}{K(r\bar{x}+K\rho)}$$

satisfies to the conditions:

$$M - \frac{p}{2} < \bar{\lambda}_2(t) < min\{\frac{p}{4}, M - \frac{c_0}{\bar{x}q}\}\,, \tag{1.12}$$

with $M = p - \frac{2rk\bar{x}}{K}(K-\bar{x})$.

Consider the example with $\bar{x} = 100000$ and $p = 20000$. Let initial size $x(0) = 150000$, the moment $t_0 = 74.95$, after that $\lambda_1(t) \equiv 0$ (fig. 2.3). Consequently, the initial control is $s^* = 0$ and then $s^* = \bar{s}(t)$ (fig. 2.4). The optimal value $E^*(t)$ is showed in fig. 2.2. In this example the size of population decreases from 150000 till 100000 (fig. 2.1).

The loss of the center is $I_1 = 0.506 \cdot 10^{11}$.

1.2. General case

If the conditions of the theorems 1-2 are not satisfied we can find it in another form.

Theorem 3. *Suppose that $\lambda_1(t)$ is a continuous function. Then the optimal control of the center has one of three forms:*

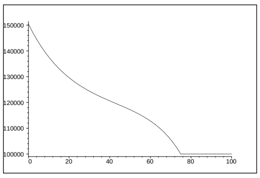

Fig. 2.1. Value $x^*(t)$

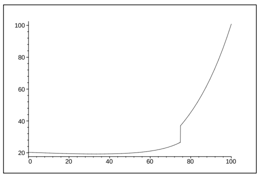

Fig. 2.2. Value $E^*(t)$

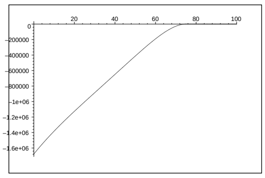

Fig. 2.3. Value $\lambda_1(t)$

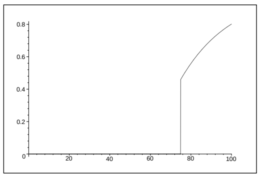

Fig. 2.4. Value $s^*(t)$

1) $s^*(t) \equiv 0,\ t \in [0,T]$,
2) $s^*(t) = \begin{cases} 1\,, & (\lambda_1(t) > 0)\ \ t < t_0\,, \\ 0\,, & (\lambda_1(t) < 0)\ \ t > t_0\,. \end{cases}$
3) $s^*(t) \equiv 1,\ t \in [0,T]$.
The optimal control of the player, respectively, is:
1) $E^*(t) = \frac{(p-\bar\lambda_2(t))qx(t)-c_0}{2kq^2x(t)^2},\ t \in [0,T]$,
2) $E^*(t) = \begin{cases} 0\,, \ \ t < t_0\,, \\ \frac{(p-\bar\lambda_2(t))qx(t)-c_0}{2kq^2x(t)^2}\,, \ \ t > t_0\,. \end{cases}$
3) $E^*(t) \equiv 0,\ t \in [0,T]$.
Sufficient conditions for the optimality are

$$x(t) > \frac{2c_0}{pq}\,, \ \ \bar\lambda_2(t) < \frac{p}{4}\,.$$

For example, let $\bar x = 200000$ and $x(0) = 150000$. Then exists the moment $t_0 = 4.5283$ in which $\lambda_1(t)$ changes sign (fig. 3.2). Consequently, at the begining $s^* = 1$, and then $s^* = 0$. Optimal value $E^*(t)$ is in fig.3.3. The size of population increases from 150000 to 230000 (fig.3.1) and the catching size is equal to 2500 (fig.3.4).

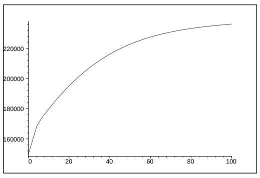

Fig. 3.1. Value $x^*(t)$

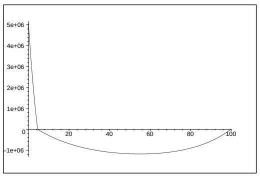

Fig. 3.2. Value $\lambda_1(t)$

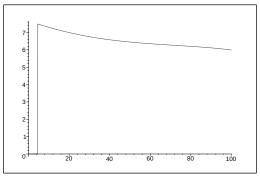

Fig. 3.3. Value $E^*(t)$

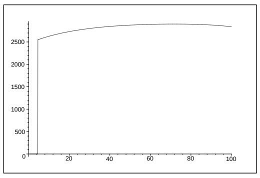

Fig. 3.4. Value $U^*(t)$

The payoff of the palyer is $J = 0.289 \cdot 10^9$, and the loss of the center is $I_1 = 0.696 \cdot 10^{11}$.

2. Another model of optimal control problem

As before, the center determines the part of the lake $s(t)$, $0 \leq s(t) \leq 1$. Time horyzon is T and the dynamics of population is:

$$x'(t) = F(x(t)) - qE(t)(1 - s(t))x(t),\ 0 \leq t \leq T,\ x(0) = x_0\,, \tag{2.1}$$

where $x(t) \geq 0$ – size of population at the moment t, $E(t) \geq 0$ – fishing efforts, $s(t)$ – part of the lake where fishery is prohibited and $q > 0$ – catching coefficient. $F(x) = rx(1 - x/K)\,.$

Payoff of the player is analogous

$$J = g(x(T)) + \int\limits_0^T e^{-\rho t}[\Pi(q, s(t), x(t), E(t)) \cdot qE(t)(1 - s(t))x(t) - c^0 E(t)]\,dt\,, \tag{2.2}$$

or

$$J = g(x(T)) + \int\limits_0^T [-\frac{1}{2}aE^2(t)(1 - s(t))^2 x^2(t) + bE(t)(1 - s(t))x(t) - cE(t)]\,dt\,,$$

where $a = 2kq^2 \exp{(-\rho t)}\,, b = pq \exp{(-\rho t)}\,, c = c^0 \exp{(-\rho t)}\,.$

Consider the new criterion for the center

$$I_2 = -\int\limits_0^T [qE(t)(1 - s(t))x(t) - \hat{x}(t)]^2 dt\,, \tag{2.3}$$

where $\hat{x}(t)$ – demand level.

Nash equilibrium solution satisfies:

$$\begin{cases} J(s^*, E^*) \geq J(s^*, E)\,,\ \forall E\,, \\ I_2(s^*, E^*) \geq I_2(s, E^*)\,,\ \forall s\,. \end{cases} \tag{2.4}$$

Fix a control function of the player. The Hamiltonian for the leader is

$$H_1 = -(qE(1 - s)x - \hat{x})^2 + \lambda_1(F(x) - qE(1 - s)x)\,.$$

Maximum H_1 is obtained in

$$s = 1 - \frac{2\hat{x} - \lambda_1}{2qEx}\,.$$

According to Maximim principle [8] we obtain the equation for λ_1

$$\lambda_1' = -\frac{\partial H}{\partial x} = 2(qE(1-s)x - \hat{x})qE(1-s) - \lambda_1(F_x'(x) - qE(1-s))\,,$$

and the transversability condition is $\lambda_1(T) = 0\,.$

Now fix a control function of the center. The Hamiltonian for the player is

$$H_2 = -\frac{1}{2}aE^2(1-s)^2x^2 + bE(1-s)x - cE + \lambda_2(F(x) - qE(1-s)x)\,.$$

Then maximum H_2 is obtained on

$$E(t) = \left[\frac{(b - q\lambda_2)(1-s)x - c}{a(1-s)^2x^2}\right]^+ .$$

It is non-negative if

$$(b - q\lambda_2)(1-s)x \geq c\,. \tag{2.5}$$

The equation for λ_2 with transversability give

$$\lambda_2' = -\frac{\partial H_2}{\partial x} = -E(1-s)(b - aE(1-s)x) - \lambda_2(F_x' - qE(1-s))\,,\ \lambda_2(T) = g_x'(x^*(T))\,.$$

Substituting s into equation for E we obtain:

$$E^*(t) = \frac{(2\hat{x} - \lambda_1(t))(2qb - 2q^2\lambda_2(t) - 2a\hat{x} + a\lambda_1(t))}{4cq^2}\,,\ 0 \leq t \leq T\,,$$

$$s^*(t) = 1 - \frac{2cq}{x^*(t)(2qb - 2q^2\lambda_2(t) - 2a\hat{x} + a\lambda_1(t))}\,.$$

Substituting it into the system we have

$$\left\{\begin{array}{rcl} x'(t) & = & rx(t)(1 - \frac{x(t)}{K}) - \frac{2\hat{x} - \lambda_1(t)}{2} \\ \lambda_1'(t) & = & -\lambda_1(t)(\frac{\hat{x}}{x(t)} - \frac{\lambda_1(t)}{2x(t)})) - \lambda_1(t)(r - \frac{2rx(t)}{K} - \frac{\hat{x}}{x(t)} + \frac{\lambda_1(t)}{2x(t)}) \\ & = & -\lambda_1(t)(r - \frac{2rx(t)}{K}) \\ \lambda_2'(t) & = & -\frac{2\hat{x} - \lambda_1(t)}{2qx(t)} + \frac{2bq - a(2\hat{x} - \lambda_1(t))}{2q} - \lambda_2(t)(r - \frac{2rx(t)}{K} - \frac{\hat{x}}{x(t)} + \frac{\lambda_1(t)}{2x(t)})\,. \end{array}\right.$$

The solution of $\lambda_1'(t) = -\lambda_1(t)(r - \frac{2rx(t)}{K})$, $\lambda_1(T) = 0$ is $\lambda_1(t) \equiv 0\,.$
Hence

$$\left\{\begin{array}{l} x'(t) = rx(t)(1 - \frac{x(t)}{K}) - \hat{x} \\ \lambda_2'(t) = -\frac{\hat{x}}{qx(t)}(b - \frac{a\hat{x}}{q}) - \lambda_2(t)(r - \frac{2rx(t)}{K} - \frac{\hat{x}}{x(t)})\,. \end{array}\right.$$

Changing arguments $\bar{\lambda}_2(t) = \lambda_2(t)e^{\rho t}$ and substituting expressions for a and b we obtain:

$$\left\{\begin{array}{l} x'(t) = rx(t)(1 - \frac{x(t)}{K}) - \hat{x} \\ \bar{\lambda}_2'(t) = -\frac{\hat{x}}{qx(t)}(p - 2k\hat{x}) - \bar{\lambda}_2(t)(r - \frac{2rx(t)}{K} - \frac{\hat{x}}{x(t)} - \rho)\,. \end{array}\right. \tag{2.6}$$

So, we obtain the necessary conditions for the following theorem.

Theorem 4. *Nash equilibrium in the problem (2.4) is*

$$E^*(t) = \frac{\hat{x}(p - \bar{\lambda}_2(t) - 2k\hat{x})}{c^0}\,, \quad s^*(t) = 1 - \frac{c^0}{x^*(t)q(p - \bar{\lambda}_2(t) - 2k\hat{x})}$$

such that $x^*(t)$, $\bar{\lambda}_2(t)$ *satisfy system*

$$\left\{\begin{array}{l} x^{*\prime}(t) = rx^*(t)(1 - \frac{x^*(t)}{K}) - \hat{x} \\ \bar{\lambda}_2'(t) = -\frac{\hat{x}}{qx^*(t)}(p - 2k\hat{x}) - \bar{\lambda}_2(t)(r - \frac{2rx^*(t)}{K} - \frac{\hat{x}}{x^*(t)} - \rho)\,, \end{array}\right.$$

and

$$\hat{x} < \frac{p}{2k}\,, \quad \bar{\lambda}_2(t) < p - 2k\hat{x}\,, \quad x^*(t) > \frac{c_0}{q(p - 2k\hat{x} - \bar{\lambda}_2(t))}. \tag{2.7}$$

Consider the example with $\hat{x} = 1500$ and $x(0) = 150000$. The optimal value $E^*(t)$ you can see in fig.4.3. You see that the optimal number of boats is approximately 9. The size of population changes from 150000 to 260000 (fig.4.1). Catching rate is 1500 (fig.4.4). The optimal size of reserved area is showed in fig.4.2, and it increases in time from 0.45 to 0.7.

The payoff of the player is $J = 0.148 \cdot 10^9$ and the loss of the center is $I_2 = 0$.

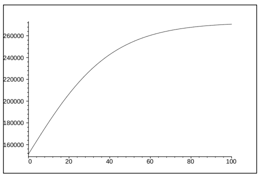

Fig. 4.1. Value $x^*(t)$

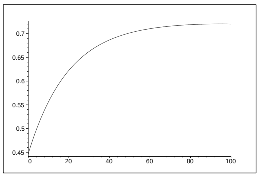

Fig. 4.2. value $s^*(t)$

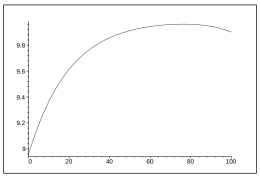

Fig. 4.3. Value $E^*(t)$

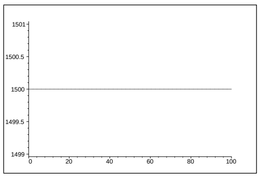

Fig. 4.4. Value $U^*(t)$

The research was supported by Russian Fund for Basic Research (project 06-01-00128-a).

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